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Conformal perturbation theory beyond the leading order

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Abstract

Higher-order conformal perturbation theory is studied for theories with and without boundaries. We identify systematically the universal quantities in the beta function equations, and we give explicit formulae for the universal coefficients at next-to-leading order in terms of integrated correlation functions. As an example, we analyse the radius dependence of the conformal dimension of some boundary operators for the case of a single Neumann brane on a circle, and for an intersecting brane configuration on a torus, reproducing in both cases the expected geometrical answer.

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1. Introduction

Perturbations of conformal field theories by relevant operators have been intensively studied starting with the work of Zamolodchikov [1, 2] on integrable perturbations of conformal field theories. Numerous examples have been considered, but there are also a number of structural results, in particular the c-theorem of Zamolodchikov [3] that states that the central charge c cannot increase along renormalization group flows, as well as the analogous g-theorem [4] for the boundary entropy [5]. Perturbations of conformal field theories also play an important role in string theory, for example, for time-dependent backgrounds; see, e.g., [6, 7].

In the context of string theory also *marginal* perturbations are of significance. Most string theories of interest possess moduli, i.e. free parameters such as the size and shape of the background or the position of some D-brane, and these correspond to marginal operators in the two-dimensional world-sheet theory. Exact conformal field theory solutions are often only available at special points (in particular, the rational points) in moduli space, and it is important to learn to control the theory away from these special points, i.e. after perturbations by marginal operators.

Usually one thinks of the moduli as corresponding to *exactly marginal* operators, and then the renormalization group analysis is, by definition, trivial. However, in the context of bulk and boundary perturbations, the situation can be more subtle. In particular, exactly marginal bulk operators (describing moduli of the closed string background) can cease to be exactly marginal in the presence of a boundary. If this is the case, they induce a non-trivial renormalization group flow on the boundary [8].

In the analysis of perturbations by relevant operators, a first-order analysis is usually sufficient (see, e.g., [9, 10]). However, in the context of marginal perturbations, it is often necessary to go to higher order in perturbation theory. The simplest example of such a situation is a single Neumann brane on a circle, for which the conformal dimension of the momentum eigenstates depends on the radius modulus. From the point of view of the world-sheet, the change in conformal dimension for this boundary operator does not arise at first order, but only appears at next-to-leading order in perturbation theory.

Higher-order conformal perturbation theory also plays a role in proofs of integrability of particular bulk and/or boundary perturbations [2, 11, 12]. Conformal perturbation theory at higher orders was studied for particular models in [13–16]; general aspects of the pure bulk case were also discussed in [17, 18].

In this paper we make an attempt at a systematic analysis of conformal perturbation theory beyond the leading order. We begin by analysing which RG coefficients are scheme independent (or universal) and thus can have a physical interpretation. (In particular, we show that this is the case for the coefficient describing the change in conformal dimension of the momentum fields on the Neumann brane.) We then outline a specific scheme—the position space minimal subtraction scheme—in which higher-order RG coefficients can be calculated. This allows us to prove that the combined bulk–boundary perturbation problem is renormalizable at the quadratic order. While the minimal subtraction scheme is conceptually clean, explicit calculations of the RG coefficients are often rather cumbersome. We therefore also consider another, Wilsonian-type scheme, to which we refer to as the 'OPE scheme' since the first non-trivial terms in the beta functions are given by OPE coefficients. This scheme has some conceptual shortcomings at higher orders but is computationally somewhat simpler. For the universal coefficients we are interested in, the result is independent of which of the two schemes we use (as we also verify explicitly). We can therefore determine the coefficients of interest (in particular, the formula for the shift in conformal dimension for the momentum fields on the Neumann brane) in the Wilsonian approach. In the resulting formulae the universal quantities are expressed as integrals over certain correlation functions. As an illustration we also apply the formulae to an intersecting brane model on a torus, and again reproduce the geometric result.

The paper is organized as follows. In section 2, we discuss which RG coefficients in the boundary beta function are universal in the presence of marginal bulk perturbations (section 2.1). We then introduce the minimal subtraction scheme, both for pure boundary perturbations (section 2.2.1) as well as for the combined bulk–boundary problem for which we prove renormalizability at the quadratic order (section 2.2.2). We also introduce the Wilsonian scheme and discuss its advantages and shortcomings (section 2.3). Finally, we explain how the discussion can be generalized to include boundary changing operators (section 2.4). In section 3 these ideas are applied to two examples, the single Neumann brane on a circle (section 3.1) as well as a configuration of two intersecting D1-branes on a 2-torus (section 3.2). Finally, we discuss in section 4 how our techniques for the calculation of higher-order RG coefficients can also be applied to pure bulk or pure boundary perturbation theory.

2. Bulk–boundary perturbations of BCFTs

Let us start with a general discussion of the effect of perturbations by marginal bulk operators on boundary degrees of freedom. Generically such a deformation will induce a renormalization group (RG) flow on the space of boundary conditions [8]. Under a certain condition, to be formulated precisely below, the induced boundary deformation is however scale independent to the first order in the bulk deformation parameter. In this case, one can study how the set of boundary scaling dimensions changes with the bulk deformation. We will demonstrate that (despite no occurrence of RG flows) the RG technique is very useful in addressing this question. In particular we will derive, using the RG methods, general expressions for a first-order change in dimensions of boundary operators along a bulk deformation.

2.1. Universal terms in marginal bulk perturbations

Before we give a detailed discussion we need to introduce some notation. Consider a boundary conformal field theory (BCFT) defined on the upper half plane $\mathbb{H}^+ = \{(x, y)|y \geqslant 0\}$ with complex coordinate $z = x + \mathrm{i}y$. Let $\phi_k(z, \bar{z})$ be bulk primary fields with conformal weights (h_k, h_k) so that their scaling dimensions are $\Delta_k = 2h_k$. For a single (fundamental) conformal boundary condition we denote the boundary primaries by $\psi_p(x)$ and their scaling dimensions by h_p. Later we will generalize our discussion to superpositions of conformal boundary conditions. We will assume that the two-point functions are normalized as

$$\langle\phi_i(z, \bar{z})\phi_j(w, \bar{w})\rangle = \frac{\delta_{ij}}{|z-w|^{2\Delta_i}}, \qquad \langle\psi_p(x)\psi_q(y)\rangle = \frac{\delta_{pq}}{|z-w|^{2h_p}}. \tag{2.1}$$

In particular, this means that we assume all fields to be self-conjugate; this is obviously not a real restriction, and our analysis can easily be generalized. The operator product expansion (OPE) for pairs of bulk and boundary operators has the form

$$\phi_i(z, \bar{z})\phi_j(w, \bar{w}) = \sum_k C_{ij}{}^k |z-w|^{\Delta_k - \Delta_i - \Delta_j}\phi_k(w, \bar{w}) + \cdots, \tag{2.2}$$

$$\psi_p(x)\psi_q(y) = \sum_r D_{pq}{}^r (y-x)^{h_r - h_p - h_q}\psi_r(y) + \cdots \quad (y > x). \tag{2.3}$$

Finally, when a bulk operator approaches the boundary it can be expanded using the bulk-to-boundary OPE

$$\phi_k(x + \mathrm{i}y, x - \mathrm{i}y) = \sum_p B_k{}^p (2y)^{h_p - \Delta_k}\psi_p + \cdots. \tag{2.4}$$

With these preparations, let us now consider a perturbation of the given BCFT generated by the Euclidean action perturbation:

$$\delta S = \sum_k l^{\Delta_k - 2}\lambda^k \iint \mathrm{d}x\,\mathrm{d}y\,\phi_k(x, y) + \sum_p l^{h_p - 1}\mu^p \int \mathrm{d}x\,\psi_p(x). \tag{2.5}$$

Here λ^k, μ^p are the dimensionless coupling constants of the respective operators, and l is a renormalization distance scale. Up to second order in the coupling constants, the beta functions have the following general form:

$$\beta^k = y_k \lambda^k + \sum_{ij} \mathcal{C}_{ij}^k \lambda^i \lambda^j + \cdots \tag{2.6}$$

$$\beta^p = y_p \mu^p + \sum_i \mathcal{B}_i^p \lambda^i + \sum_{qr} \mathcal{D}_{qr}^p \mu^q \mu^r + \sum_{iq} \mathcal{E}_{iq}^p \lambda^i \mu^q + \cdots . \tag{2.7}$$

Here, $y_k = 2 - \Delta_k$ and $y_q = 1 - h_q$ are the bulk and boundary anomalous dimensions, respectively. The omitted terms stand for higher orders in the coupling constants. A general property of any local RG scheme is that the bulk beta functions are independent of the boundary couplings.

It has been known for quite some time [19] (see also [20]) that in a particular renormalization scheme the coefficients $\mathcal{C}_{ij}^k$ for bulk theories are given by $\mathcal{C}_{ij}^k = \pi C_{ij}{}^k$, where $C_{ij}{}^k$ are the bulk OPE coefficients from (2.2). The same scheme, to be discussed in more detail in section 2.3, can easily be adapted for theories on the half plane. The coefficients $\mathcal{D}_{qr}^p$ coincide then with the boundary structure constants $D_{qr}{}^p$ (see, e.g., [21]), and for the coefficients $\mathcal{B}_i^p$ we have $\mathcal{B}_i^p = \frac{1}{2} B_i{}^p$, where $B_i{}^p$ are the bulk-to-boundary OPE coefficients from (2.4); see [8].

Consider now the case where we perturb the BCFT by a single bulk field $\phi(x, y)$ with a coupling constant λ. Furthermore, we want to assume that the bulk beta function $\beta^\phi(\lambda)$ vanishes. In this case, even in the absence of an initial boundary perturbation $\mu_{\text{bare}}^p = 0$, a boundary renormalization group flow can be triggered by the terms $\mathcal{B}_\phi^p \lambda$ in the boundary beta function. Such boundary terms, however, are in general not universal. For example, if the induced boundary fields are all relevant, i.e. $y_p > 0$, then the corresponding terms in the boundary beta function can be removed by a coupling constant redefinition

$$\mu^p \mapsto \tilde{\mu}^p = \mu^p + \frac{\mathcal{B}_\phi^p}{y_p} \lambda. \tag{2.8}$$

The above coupling constant redefinition looks peculiar in that $\tilde{\mu}^p$ is not proportional to μ^p. It has, however, a simple meaning. Let $Z = \langle e^{\delta S} \rangle$ be the renormalized partition function[4] of the perturbed theory (2.5). We have

$$\left(\frac{\partial \ln Z}{\partial \lambda}\right)_{\{\tilde{\mu}^p\}} - \left(\frac{\partial \ln Z}{\partial \lambda}\right)_{\{\mu^p\}} = -\sum_p \frac{\mathcal{B}_\phi^p}{y_p} \left(\frac{\partial \ln Z}{\partial \tilde{\mu}^p}\right)_\lambda = -\sum_p \frac{\mathcal{B}_\phi^p}{y_p} \left(\frac{\partial \ln Z}{\partial \mu^p}\right)_\lambda , \tag{2.9}$$

where the partial derivatives on the left-hand side are taken with the boundary constants μ^p or $\tilde{\mu}^p$ held fixed. The identities (2.9) mean that, after the redefinition (2.8), the bulk coupling constant λ couples to a re-defined field

$$\tilde{\phi}(x, y) = \phi(x, y) - \sum_p \frac{\mathcal{B}_\phi^p}{y_p} \psi_p(x) \delta(y). \tag{2.10}$$

Part of the renormalization procedure amounts to defining the operator coupling to λ on the half plane so that the correlation functions involving that operator are distributions (and thus integrable in any bounded region on the half plane). In the interior of the half plane the resulting operator must coincide with the bulk operator $\phi(x, y)$, but in general extra subtractions may be required at the boundary. The redefinition (2.10) stemming from the change of scheme (2.8) reflects the natural ambiguity in defining such a fully subtracted operator extending $\phi(x, y)$.

[4] Here we are talking about the partition function on a disc. The passage from the half plane to the disc is straightforward because the bulk theory is conformal.

Suppose now that all terms linear in λ can be removed in this manner. Then the resulting boundary beta functions have the form

$$\tilde{\beta}^p = \sum_q D^p_q(\lambda)\tilde{\mu}^q + \mathcal{O}(\tilde{\mu}^2), \tag{2.11}$$

where

$$D^p_q(\lambda) = y_p\delta^p_q + \lambda\tilde{\mathcal{E}}^p_{\phi q} + \mathcal{O}(\lambda^2) \qquad \text{with} \quad \tilde{\mathcal{E}}^p_{\phi q} = \mathcal{E}^p_{\phi q} - 2\sum_r \mathcal{D}^p_{(qr)}\frac{\mathcal{B}^r_\phi}{y_r} \tag{2.12}$$

and $\mathcal{D}^p_{(qr)} = \frac{1}{2}\big(\mathcal{D}^p_{qr} + \mathcal{D}^p_{rq}\big)$. Now that the boundary beta functions are all proportional to the boundary coupling constants we can treat the boundary perturbations (at least those of the relevant operators) infinitesimally to read off the dimensions of the boundary operators in the deformed theory. More specifically, we claim that the eigenvalues of the matrix $D^p_q(\lambda)$ are to be identified with $y = 1 - h$, where h is the scaling dimension of the boundary operator in the deformed theory, and the corresponding boundary primaries are the eigenvectors of $D^p_q(\lambda)$. To leading order in λ, the matrix $D^p_q(\lambda)$ can be diagonalized by the transformation

$$\tilde{\mu}^p \mapsto \sum_q \big(\delta^p_q + \lambda f^p_q\big)\tilde{\mu}^q, \qquad \text{where} \quad f^p_q = \begin{cases} \dfrac{\tilde{\mathcal{E}}^p_{\phi q}}{y_p - y_q} & \text{for} \quad p \neq q \\ 0 & \text{for} \quad p = q. \end{cases} \tag{2.13}$$

The corresponding primary fields are[5]

$$\psi_p[\lambda] = \psi_p - \lambda \sum_{q\neq p} \frac{\tilde{\mathcal{E}}^q_{\phi p}}{y_q - y_p}\psi_q, \tag{2.14}$$

and their anomalous dimensions are

$$y_p[\lambda] = y_p + \lambda\tilde{\mathcal{E}}^p_{\phi p}. \tag{2.15}$$

We further claim that the quantity specifying the dimension shifts

$$\tilde{\mathcal{E}}^p_{\phi p} = \mathcal{E}^p_{\phi p} - 2\sum_r \mathcal{D}^p_{(pr)}\frac{\mathcal{B}^r_\phi}{y_r} \tag{2.16}$$

is scheme independent. To see this we consider a coupling constant redefinition of the form

$$\mu^p \mapsto \mu^p + \lambda b^p + \sum_{qr} d^p_{qr}\mu^q\mu^r + \sum_q e^p_q\lambda\mu^q + \cdots. \tag{2.17}$$

Under this redefinition the coefficients in the beta functions (2.7) change as

$$\begin{aligned} \mathcal{B}^p_\phi &\mapsto \mathcal{B}^p_\phi - b^p y_p, \\ \mathcal{D}^p_{rs} &\mapsto \mathcal{D}^p_{rs} + d^p_{(rs)}(y_r + y_s - y_p), \\ \mathcal{E}^p_{\phi r} &\mapsto \mathcal{E}^p_{\phi r} - 2\sum_s \mathcal{D}^p_{rs}b^s - 2\sum_s d^p_{(rs)}b^s(y_r + y_s - y_p) + 2\sum_s d^p_{(rs)}\mathcal{B}^s_\phi + e^p_r(y_r - y_p). \end{aligned} \tag{2.18}$$

It is straightforward to check that under the transformations (2.18) the quantity (2.16) is indeed invariant.

[5] The fields $\psi_p[\lambda]$ are defined up to adding a multiple of $\lambda\psi_p$.

2.2. Computation in a minimal subtraction scheme

In the following we want to explain in detail how these coefficients—in particular (2.16)—can be calculated explicitly. We shall first study this question in a minimal subtraction scheme. In order to make sense of the formal perturbation series we shall use a point-splitting regularization. In particular, we require that any two perturbing bulk or boundary fields do not approach each other closer than a cut-off ϵ, and that the perturbing bulk fields only approach the boundary up to a distance $\epsilon/2$. Before specializing to the bulk and boundary situation discussed in the previous section, let us first discuss some generalities of renormalization. For brevity we consider only boundary perturbations but that are inessential for the points we want to make.

2.2.1. Generalities of minimal subtraction schemes for boundary perturbations. Let us consider a perturbed BCFT action

$$S = S_{\mathrm{BCFT}} + \sum_p \mu_{\mathrm{B}}^p \int \mathrm{d}x \, \psi_p(x), \tag{2.19}$$

where μ_{B}^p are the bare coupling constants. Let l be an infrared distance scale at which we wish to renormalize the theory. In terms of the renormalized dimensionless coupling constants μ^p, the same Lagrangian (2.19) can be expressed as

$$S = S_{\mathrm{BCFT}} + \sum_p l^{-y_p} \mu^p \int \mathrm{d}x \, \psi_p(x) + S_{\mathrm{ct}}, \tag{2.20}$$

where y_p are anomalous dimensions of the fields $\psi_p(x)$, and S_{ct} is a counterterm action. Perturbation theory generates integrals of the form

$$\int \cdots \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \psi_{p_1}(x_1)\psi_{p_2}(x_2) \cdots \psi_{p_n}(x_n) \prod_{i<j}^{n} \theta(|x_i - x_j| - \epsilon)\theta(L - |x_i - x_j|), \tag{2.21}$$

where we have also introduced an infrared regulator L. The above expression is to be understood in the operator sense, i.e. inside a correlator with arbitrary other insertions. The product of fields in (2.21) can be expanded in terms of a complete set of local operators Ψ_A as

$$\psi_{p_1}(x_1)\psi_{p_2}(x_2) \ldots \psi_{p_n}(x_n) = \sum_A C^A_{p_1,\ldots,p_n}(x_1, \ldots, x_{n-1})\Psi_A(x_n), \tag{2.22}$$

where we have arbitrarily chosen the point of insertion on the right-hand side to be x_n. If the OPEs of the conformal families of the primaries ψ_p close on themselves, we can take for Ψ_A the fields ψ_p and their conformal descendants. In conformal field theory the expansion (2.22) always converges [22] unlike in massive QFTs for which the OPE may be merely an asymptotic expansion. Substituting (2.22) into (2.21) we obtain expressions of the form

$$\sum_A C^A_{p_1,\ldots,p_n}(\epsilon, L) \int \mathrm{d}x \, \Psi_A(x), \tag{2.23}$$

where

$$\begin{aligned} C^A_{p_1,\ldots,p_n}(\epsilon, L) = {} & \int \cdots \int \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} C^A_{p_1,\ldots,p_n}(x_1, \ldots, x_{n-1}) \\ & \times \prod_{i<j}^{n} \theta(|x_i - x_j| - \epsilon)\theta(L - |x_i - x_j|). \end{aligned} \tag{2.24}$$

The integrals (2.24) are finite because of the cut-offs. In the limit $\epsilon \to 0$, the coefficients $C^{A}_{p_1,\ldots,p_n}(\epsilon, L)$ diverge with the divergences coming from the regions of integration in which two or more insertion points $x_1, \ldots, x_n$ collide. In fact, if k operators $\psi_{p_1}, \ldots, \psi_{p_k}$ come close together (with the other insertions bounded away from the point of coincidence) to produce an operator Ψ_S, the leading divergence has the form

$$C\epsilon^{y_{p_1}+\cdots+y_{p_k}-y_S} L^{y_S+y_{p_{k+1}}+\cdots+y_{p_n}-y_A}, \tag{2.25}$$

where C is some numerical constant. Here we have assumed that the resonance condition,

$$y_{p_1} + \cdots + y_{p_k} - y_S = 0, \tag{2.26}$$

does not hold; otherwise the corresponding divergence is logarithmic. The above reasoning follows essentially from dimensional counting, as well as from the locality of the OPE, ensuring the independence of the expansion (2.22) from L. If we only perturb by marginal or relevant fields, $y_{p_j} \geqslant 0$, then divergences can only occur if also Ψ_S is relevant, i.e. $y_S > 0$. Assuming that the OPE is closed, Ψ_S is then one of the perturbing relevant or marginal primary fields ψ_p. In the 'minimal subtraction scheme' we are using here, we only introduce counterterms for actually divergent contributions; the above reasoning then implies that the scheme closes on itself.

The divergences arising when k operators come together first emerge at order $n = k$ in perturbation theory. One expects that they can be cancelled by local (L-independent) counterterms. These counterterms then also cancel the non-local subdivergences (2.25) that appear at order $m > k$ in perturbation theory. Thus we only need to deal with the case when $n = k$, in which case Ψ_S and Ψ_A must have a non-trivial two-point function, and hence $y_S = y_A$. Then the coefficient (2.25) is independent of L, and hence converges when $L \to \infty$. Note that the lower-order counterterms may also contribute to the $k = n$ divergence when the counterterm insertion from the order $l < n$ comes close together with $n - l$ fields ψ_{p_i}. The same dimensional reasoning however tells us that the final coefficient must again be independent of L, and the remaining divergence can be cancelled by a local counterterm.

The above discussion should however not be taken to be a recursive proof of renormalizability of conformal perturbation theory. One problem that needs to be tackled is the classical problem of overlapping divergences; in the case at hand this occurs when k points come together with a subset of $l < k$ points coming together much faster than the remaining ones. The associativity of the OPE in conformal field theory should be the key property ensuring the consistency in dealing with overlapping divergences, but we have not attempted to work this out in detail. However, we will see in the concrete examples of the next subsections how the above discussion can be made more rigorous. In particular, we will prove the renormalizability of conformal perturbation theory at the next-to-leading order using analytic properties of conformal blocks.

In order to illustrate these ideas, let us now consider an integral that emerges at second order in perturbation theory,

$$\frac{1}{2!}\sum_{p,q} l^{-y_p-y_q}\mu^p\mu^q \int \mathrm{d}x_1 \int \mathrm{d}x_2\, \psi_p(x_1)\psi_q(x_2)\theta(|x_1-x_2|-\epsilon)\theta(L-|x_1-x_2|). \tag{2.27}$$

The product of the two boundary fields can be expanded via the OPE (2.3). Performing one of the integrals, we see that we get ultraviolet divergences of the form (in the limit $L \to \infty$)

$$S^{(2)}_{\mathrm{div}} = -\frac{1}{2}\sum_{p,q,r} \frac{D_{pq}{}^{r}}{y_p+y_q-y_r}\left(\frac{\epsilon}{l}\right)^{y_p+y_q-y_r} l^{-y_r}\mu^p\mu^q \int \mathrm{d}x\, \psi_r(x), \tag{2.28}$$

where the summation runs only over those indices p, q, r for which $y_p + y_q - y_r < 0$. In particular, $y_r > 0$, and thus only relevant primary fields ψ_r contribute. As before, we have also assumed here that there are 'no resonances', i.e. that none of the expressions $y_p + y_q - y_r$ vanishes. Then only power divergences occur at this order.

The above divergences can be cancelled by adding a minimal action counterterm $S_{\text{ct}}^{(2)} = -S_{\text{div}}^{(2)}$. Equating the two expressions (2.19) and (2.20) we obtain up to second order in the coupling constants

$$\mu_{\text{B}}^r = l^{-y_r}\left[\mu^r + \frac{1}{2}\sum_{p,q\in I_r^{(2)}} \frac{D_{pq}{}^r}{y_p + y_q - y_r}\left(\frac{\epsilon}{l}\right)^{y_p+y_q-y_r} l^{-y_r}\mu^p\mu^q\right], \tag{2.29}$$

where $I_r^{(2)}$ is the set of pairs of indices (p, q) for which $y_p + y_q - y_r < 0$. Differentiating both sides of (2.29) with respect to l with fixed μ_B^r, we obtain

$$l\frac{\mathrm{d}\mu^r}{\mathrm{d}l} = \beta^r(\mu) = y_r\mu^r. \tag{2.30}$$

Thus the beta functions are linear in μ. It is easy to see that this property continues to hold also at higher order in perturbation theory, as long as the divergences are power like.

On the other hand, if we have a non-trivial resonance at lowest order, i.e. if $y_r = y_p + y_q$, then formula (2.28) takes the form (we are assuming for simplicity that there are no other divergences at this order)

$$S_{\text{div}}^{(2)} = -\frac{1}{2}D_{pq}{}^r\ln(\epsilon/l)l^{-y_r}\mu^q\mu^p\int \mathrm{d}x\,\psi_r(x), \tag{2.31}$$

where we cut off the divergent integral in the infrared region at the renormalization scale l. Introducing a counterterm $S_{\text{ct}}^{(2)} = -S_{\text{div}}^{(2)}$ we then obtain a beta function for the coupling μ^r:

$$\beta^r = y_r\mu^r + D_{pq}{}^r\mu^p\mu^q. \tag{2.32}$$

More generally, in the minimal subtraction scheme at hand, the nonlinear terms in the beta functions all come from resonances. However, in general not all resonant terms are universal.

2.2.2. Minimal subtraction scheme for bulk–boundary perturbations. After this interlude we now return to the case of interest, namely the description of the minimal subtraction scheme for bulk–boundary perturbations. In fact, the above discussion generalizes in a straightforward manner to include an additional perturbation by a bulk field. For simplicity of presentation, we shall assume that the bulk field $\phi(z,\bar{z})$ is a spinless relevant or marginal primary field of scaling dimension $\Delta = 2 - y_\phi$ with $y_\phi \geqslant 0$. As we shall explain below, the bulk–boundary perturbation at the next-to-leading order in perturbation theory is then renormalizable. We will specialize to the situation where the bulk field is marginal ($y_\phi = 0$) later.

At the linear order in the bulk coupling λ the divergences in perturbation theory only arise from singularities as the bulk field approaches the boundary. These are described by the bulk-to-boundary OPE. If there are no boundary fields for which $h_p = \Delta - 1$ we have power divergences of the form (2.4)

$$S_{\text{div}}^{(1)} = -\lambda\sum_{p\in I^{(1)}} \frac{B_\phi{}^p}{2(y_\phi - y_p)}\left(\frac{\epsilon}{l}\right)^{y_\phi - y_p} l^{-y_p}\int \mathrm{d}x\,\psi_p(x), \tag{2.33}$$

where

$$I^{(1)} = \{p|y_p > y_\phi\}. \tag{2.34}$$

In the minimal subtraction scheme the counterterm is then simply $S_{\text{ct}}^{(1)} = -S_{\text{div}}^{(1)}$. If there is a boundary field for which $B_\phi{}^p \neq 0$ and the resonance condition $y_p = y_\phi$ is satisfied, we have a logarithmic divergence which results in a universal term linear in λ in the boundary beta function [8],

$$\beta^p = y_p\mu^p + \lambda\frac{B_\phi{}^p}{2} + \cdots. \tag{2.35}$$

In the case when the bulk perturbation is marginal the resonance condition requires that the boundary field is also marginal.

We shall, in the following, always assume that the resonance condition is not satisfied, i.e. that $y_p \neq y_\phi$; this is, for example, true in the context of section 2.1 where $y_\phi = 0$ and $y_p > 0$. Then the counterterm is given by $S^{(1)}_{\mathrm{ct}} = -S^{(1)}_{\mathrm{div}}$. At the next order in perturbation theory we encounter the integral

$$\sum_q \lambda\mu^q l^{-y_q - y_\phi}\int \mathrm{d}x'\left[\iint \mathrm{d}x\,\mathrm{d}y\,\theta(y-\epsilon/2)\theta(R^2-(x-x')^2-y^2)\phi(z,\bar{z})\psi_q(x')\right], \tag{2.36}$$

where $z = x + \mathrm{i}y$, and R is an infrared regulator. The quantity in the square brackets in (2.36) can be expanded in local boundary fields as in (2.22) and (2.23); the coefficients of these fields can be expressed in terms of certain integrals (see below). By the same arguments as above, only coefficients of (primary) relevant fields can be divergent as we send $\epsilon \to 0$. More precisely, the coefficient with which the primary field ψ_p will appear in (2.36) equals

$$I^p_q = \iint \mathrm{d}x\,\mathrm{d}y\,\theta(y-\epsilon/2)\theta(R^2-(x-x')^2-y^2)\langle\phi(z,\bar{z})\psi_q(0)\psi_p(\infty)\rangle. \tag{2.37}$$

Using the Möbius symmetry, the correlation function appearing in this formula can be written as

$$\langle\phi(z,\bar{z})\psi_q(0)\psi_p(\infty)\rangle = |z|^{y_q-y_p+y_\phi-2}\eta^{\delta+y_\phi-2}(1-\eta)^{(2-y_\phi-\delta)/2}Y(\eta), \tag{2.38}$$

where

$$\eta = 1-\frac{\bar{z}}{z}, \qquad \delta = \frac{1}{3}(4-y_p-y_q-y_\phi) = \frac{1}{3}(h_p+h_q+\Delta), \tag{2.39}$$

and

$$Y(\eta) = \begin{cases} \sum_A B_\phi{}^A D_{qA}{}^p\, \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(y_A-y_\phi+1)}F^A_{\phi\bar{\phi}qp}(\eta) & (\mathrm{Re}\, z>0),\\ \sum_A B_\phi{}^A D_{Aq}{}^p\, \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(y_\phi-y_A-1)}F^A_{\phi\bar{\phi}qp}(\eta) & (\mathrm{Re}\, z<0). \end{cases} \tag{2.40}$$

Here the index A runs over all conformal primaries whose conformal families appear in the intermediate channel. The conformal blocks $F^A_{\phi\bar{\phi}qp}(\eta)$ have a branch cut along the real η-axis from $-\infty$ to 1 and are normalized so that $F^A(\eta) \sim \eta^{h_A-\delta}$ with coefficient 1 as $\eta \to 0$. The conformal blocks entering the function $Y(\eta)$ are defined on opposite sides of the branch cut for $\mathrm{Re}\, z > 0$ and $\mathrm{Re}\, z < 0$. The analyticity in z implies that the values of $Y(\eta)$ in the lower half η-plane are obtained by the analytic continuation in a clockwise direction from the upper half plane [23]. Passing to polar coordinates $z = r\,\mathrm{e}^{\mathrm{i}\vartheta}$ and using (2.38) we can rewrite (2.37) as

$$I^p_q = \int_{\vartheta_*}^{\pi-\vartheta_*}\mathrm{d}\vartheta\int_{r_*(\eta)}^{R}\mathrm{d}r\, r^{y_q-y_p+y_\phi-1}\eta^{\delta+y_\phi-2}(1-\eta)^{(2-y_\phi-\delta)/2}Y(\eta), \tag{2.41}$$

where

$$\vartheta_* = \arcsin\left(\frac{\epsilon}{2R}\right), \qquad r^2_*(\eta) = \frac{\epsilon^2(\eta-1)}{\eta^2}. \tag{2.42}$$

Since $\eta = 1 - \mathrm{e}^{-2\mathrm{i}\vartheta}$ depends only on ϑ we can perform the integral over r. In the remaining integral it is convenient to change the integration variable ϑ to η. Altogether we then obtain

$$I_q^p = \frac{\mathrm{i}}{2\zeta_{pq}} \int_{C(\epsilon/R)} \mathrm{d}\eta \left[\epsilon^{\zeta_{pq}} \left(\frac{\eta - 1}{\eta^2} \right)^{\zeta_{pq}/2} - R^{\zeta_{pq}} \right] \eta^{\delta + y_\phi - 2} (1-\eta)^{-(y_\phi + \delta)/2} Y(\eta), \tag{2.43}$$

where

$$\zeta_{pq} = y_q - y_p + y_\phi, \tag{2.44}$$

and the contour of η-integration is a segment of the circle of radius 1 centred at $\eta = 1$ and oriented clockwise

$$C(\epsilon/R) = \{\eta = 1 - \mathrm{e}^{-2\mathrm{i}\vartheta}, \vartheta_* \leqslant \vartheta \leqslant \pi - \vartheta_*\}. \tag{2.45}$$

Obviously, this expression only makes sense if $\zeta_{pq} \neq 0$. In the resonance case (i.e. for $\zeta_{pq} = 0$) we have instead

$$\left(I_q^p\right)_{\mathrm{res}} = \frac{\mathrm{i}}{2} \int_{C(\epsilon/R)} \mathrm{d}\eta \ln\left(\frac{\epsilon}{R|\eta|} \right) \eta^{\delta + y_\phi - 2} (1-\eta)^{-(y_\phi+\delta)/2} Y(\eta). \tag{2.46}$$

The divergences of I_q^p and $\left(I_q^p\right)_{\mathrm{res}}$ in the limit $\epsilon \to 0$ can now be analysed using well-known properties of conformal blocks. There are two kinds of divergences that will be important to us: those that come from the region of integration $\eta \sim \epsilon/R \to 0$ where the bulk operator approaches the boundary far away from the point of insertion of ψ_q, and those that arise when the bulk field approaches the boundary in the vicinity of the boundary field ψ_q. As we have argued before (and as will become clear below) the former divergences are cancelled by the contribution from the lower-order counterterm $S_{\mathrm{ct}}^{(1)}$, while the remaining divergences have the power $\epsilon^{\zeta_{pq}}$. To see this, we use the asymptotics—see (2.40)

$$\begin{aligned} &\text{for} \quad \vartheta \to 0 \qquad Y(\eta) \sim \sum_A B_\phi{}^A D_{qA}{}^p \, \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(y_A - y_\phi + 1)} \eta^{1 - y_A - \delta} + \cdots, \\ &\text{for} \quad \vartheta \to \pi \qquad Y(\eta) \sim \sum_A B_\phi{}^A D_{Aq}{}^p \, \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(y_\phi - y_A - 1)} \eta^{1 - y_A - \delta} + \cdots, \end{aligned} \tag{2.47}$$

in (2.43) and (2.46), and then perform the η-integrals in the vicinity of $\eta = 0$, i.e. from $\eta = \mathrm{i}\frac{\epsilon}{R}$ up to some intermediate cut-off ξ. This leads to

$$I_q^p = C_q^p \epsilon^{\zeta_{pq}} + f_q^p R^{\zeta_{pq}} - \sum_A \left[\frac{B_\phi{}^A \left(D_{qA}{}^p + D_{Aq}{}^p\right)}{2(y_\phi - y_A)(y_A + y_q - y_p)} \epsilon^{y_\phi - y_A} R^{y_A + y_q - y_p} + \mathcal{O}(\epsilon^{y_\phi - y_A + 1}) \right], \tag{2.48}$$

$$\left(I_q^p\right)_{\mathrm{res}} = \left(C_q^p\right)_{\mathrm{res}} \ln(\epsilon/l) + \left(f_q^p\right)_{\mathrm{res}} + \sum_A \left[\frac{B_\phi{}^A (D_{qA}{}^p + D_{Aq}{}^p)}{2(y_\phi - y_A)^2} \left(\frac{\epsilon}{R}\right)^{y_\phi - y_A} + \mathcal{O}(\epsilon^{y_\phi - y_A + 1}) \right], \tag{2.49}$$

where C_q^p, $\left(C_q^p\right)_{\mathrm{res}}$, f_q^p and $\left(f_q^p\right)_{\mathrm{res}}$ are some constants independent of ϵ and R (that come from the evaluation of the primitive function at ξ, as well as from the remaining part of the integral). Since we are only interested in divergent contributions in ϵ, we may restrict the fields A to be relevant primary fields in $A \in I^{(1)}$. Furthermore, we can ignore all the subleading terms $\mathcal{O}(\epsilon^{y_\phi - y_A + 1})$ since they vanish in the limit $\epsilon \to 0$.[6]

[6] This is obvious for $y_A < 1$. In a unitary BCFT, $y_A = 1$ corresponds always to the identity operator Ω which does not have a descendant operator at level one since $L_{-1}\Omega = 0$.

Now we want to show that the divergent terms in the sum in (2.48) are precisely cancelled by the lower-order counterterm $S_{\mathrm{ct}}^{(1)}$ given in (2.33). At order $\lambda\mu^q$—recall that μ^q is the coupling constant corresponding to ψ_q—the counterterm leads to the contribution

$$\lambda\mu^q l^{-y_q-y_\phi}\sum_{s\in I^{(1)}}\epsilon^{y_\phi-y_s}\frac{B_\phi{}^s}{2(y_\phi-y_s)}\iint \mathrm{d}x\,\mathrm{d}x'\,\theta(|x-x'|-\epsilon)\theta(R-|x-x'|)\psi_s(x)\psi_q(x'). \tag{2.50}$$

Again this can be expanded in terms of local boundary fields, and the divergence in the coefficient of ψ_p equals

$$\left(I_{ct}^{(1)}\right)_q^p=\sum_{s\in I^{(1)}}\frac{B_\phi^s\left(D_{qs}{}^p+D_{sq}{}^p\right)}{2(y_\phi-y_s)(y_s+y_q-y_p)}\left[\epsilon^{y_\phi-y_s}R^{y_s+y_q-y_p}-\epsilon^{\zeta_{pq}}\right]. \tag{2.51}$$

Here we have assumed that there is no resonance among the boundary fields, i.e. that $y_s+y_q\neq y_p$ for any $s\in I^{(1)}$; if there is a resonance, i.e. $y_s+y_q=y_p$, then (2.51) has to be modified in the obvious manner.

In either case, by comparison with (2.48), it is now clear that the contribution $\left(I_{ct}^{(1)}\right)_q^p$ cancels precisely the divergent part of the sum in (2.48), and similarly for the resonant case $\zeta_{pq}=0$. Thus the divergent contribution only comes from the first term in (2.48) and (2.49):

$$\begin{aligned}\tilde I_q^p&=I_q^p+\left(I_{ct}^{(1)}\right)_q^p\sim\tilde C_q^p\epsilon^{\zeta_{pq}},\\ \left(\tilde I_q^p\right)_{\mathrm{res}}&=\left(I_q^p\right)_{\mathrm{res}}+\left(\left(I_{ct}^{(1)}\right)_q^p\right)_{\mathrm{res}}\sim\left(\tilde C_q^p\right)_{\mathrm{res}}\ln(\epsilon/l)\end{aligned}\qquad \text{as}\quad \epsilon\to0, \tag{2.52}$$

where $\tilde C_q^p$ and $\left(\tilde C_q^p\right)_{\mathrm{res}}$ are coefficients that can be obtained by taking the limits

$$\tilde C_q^p=\lim_{\epsilon\to0}\epsilon^{-\zeta_{pq}}\epsilon\partial_\epsilon\tilde I_q^p,\qquad \left(\tilde C_q^p\right)_{\mathrm{res}}=\lim_{\epsilon\to0}\epsilon\partial_\epsilon\left(\tilde I_q^p\right)_{\mathrm{res}}. \tag{2.53}$$

Using the explicit expressions (2.43), (2.46) and (2.51) we finally obtain

$$\begin{aligned}\tilde C_q^p=\lim_{\epsilon\to0}&\left[\frac{\mathrm{i}}{2\zeta_{pq}}\int_{C(\epsilon/R)}\mathrm{d}\eta(1-\eta)^{-(y_\phi+\delta)/2}(\eta-1)^{\zeta_{pq}/2}\eta^{y_p-y_q+\delta-2}Y(\eta)\right.\\ &\left.+\sum_{s\in I^{(1)}}\frac{B_\phi{}^s\left(D_{qs}{}^p+D_{sq}{}^p\right)}{2(y_s+y_q-y_p)\zeta_{pq}}\left(\frac{R}{\epsilon}\right)^{y_s+y_q-y_p}\right]+\sum_{s\in I^{(1)}}\frac{B_\phi{}^s\left(D_{qs}{}^p+D_{sq}{}^p\right)}{2(y_s-y_\phi)(y_s+y_q-y_p)},\end{aligned} \tag{2.54}$$

$$\begin{aligned}\left(\tilde C_q^p\right)_{\mathrm{res}}=\lim_{\epsilon\to0}&\left[\frac{\mathrm{i}}{2}\int_{C(\epsilon/R)}\mathrm{d}\eta\,\eta^{y_\phi+\delta-2}(1-\eta)^{-(y_\phi+\delta)/2}Y(\eta)\right.\\ &\left.+\sum_{s\in I^{(1)}}\frac{B_\phi{}^s\left(D_{qs}{}^p+D_{sq}{}^p\right)}{2(y_s-y_\phi)}\left(\frac{R}{\epsilon}\right)^{y_s-y_\phi}\right].\end{aligned} \tag{2.55}$$

It is worth noting that the contours of the η-integrations in (2.54) and (2.55) can be deformed provided the ends of the contour are held fixed and the cut is not crossed. In particular, one can deform $C(\epsilon/R)$ to run infinitesimally above the cut to $\eta=1$, and then infinitesimally below back to the second endpoint near $\eta=0$.

Given that the conformal blocks only have singularities at $\eta=1,0,\infty$ with standard asymptotics, the quantities $\tilde C_q^p$, $\left(\tilde C_q^p\right)_{\mathrm{res}}$ defined in (2.54) and (2.55) are finite and independent of R. This essentially provides a proof that at order $\lambda\mu^p$ the divergences can be cancelled by local counterterms which are linear combinations of relevant operators. Together with the well-known results at orders λ^2 and $\mu^p\mu^q$ we have thus shown that a generic perturbation by relevant or marginal bulk and boundary fields is renormalizable at the quadratic order in the couplings[7]. It should also be possible to extend the analysis to higher orders in perturbation theory, but we have not attempted to do so.

If the resonance condition $\zeta_{pq} = 0$ is not satisfied the divergence of $\tilde{I}_q^p$ is power like, and in the minimal subtraction scheme there are no terms of order $\lambda\mu^p$ in the beta function β^p. On the other hand, when the resonance condition is satisfied for a pair (p, q), we have a universal term (cf (2.31) and (2.32))

$$\beta^p = y_p\mu^p - \left(\tilde{C}_q^p\right)_{\rm res}\lambda\mu^q + \cdots. \tag{2.56}$$

For a marginal bulk perturbation $y_\phi = 0$, the $p = q$ case (and only that one) is always resonant so that we have in the notation of section 2.1

$$\mathcal{E}_{\phi q}^p = \left(\mathcal{E}_{\phi q}^p\right)_{\min} \equiv \begin{cases} -\left(\tilde{C}_p^p\right)_{\rm res} & \text{for} \quad p = q \\ 0 & \text{for} \quad p \neq q. \end{cases} \tag{2.57}$$

Moreover, assuming as in section 2.1 that all $y_p > 0$, we have in the minimal subtraction scheme

$$\mathcal{D}_{pr}^p = \mathcal{D}_{rp}^p = 0, \qquad \mathcal{B}_\phi^p = 0, \tag{2.58}$$

and therefore by (2.16)

$$\tilde{\mathcal{E}}_{\phi p}^p = -\left(\tilde{C}_p^p\right)_{\rm res}, \tag{2.59}$$

where $\left(\tilde{C}_p^p\right)_{\rm res}$ is given by formula (2.55) for $y_\phi = 0$ and $p = q$. As we proved in section 2.1, the quantity $\tilde{\mathcal{E}}_q^p$ is scheme independent; we have therefore managed to obtain a description of this universal quantity in terms of conformal blocks—this is the main result of this subsection.

2.3. *Computation in a Wilsonian scheme*

It is instructive to compute (2.16) also in a different, Wilsonian-type, renormalization scheme—that of [8, 20, 21]. We will refer to this scheme as the 'OPE scheme' for the reason that the first non-trivial terms in the beta functions are given by various OPE coefficients. This scheme is often employed in conformal perturbation theory at the leading order. One of the advantages of this scheme is that in the presence of a nearby infrared fixed point in theory space, the corresponding coordinates are nonsingular near that fixed point. Another attractive feature is that formulae for universal quantities, such as the dimension shift (2.16), can be obtained quite easily in contrast to the minimal subtraction scheme. On the other hand, we will see at the end of this section that the scheme has some pitfalls when applied to computing non-universal quantities at higher order in perturbation theory.

In the OPE scheme the theory is also regulated by a point-splitting cut-off ϵ, just as in the minimal subtraction scheme of the previous section. It is however convenient to introduce the infrared regulator slightly differently: we introduce a cut-off whenever two coordinates x_i, x_j are separated by a distance larger than L, and whenever there is a bulk operator at a distance $y > L$. The dimensionless couplings are now introduced by using the UV cut-off scale itself, which is understood as a fundamental UV scale (lattice spacing, atomic or molecular scale). Thus we have

$$\delta S = \sum_k \epsilon^{\Delta_k - 2}\lambda^k \iint \mathrm{d}x\,\mathrm{d}y\,\phi_k(x, y) + \sum_p \epsilon^{h_p - 1}\mu^p \int \mathrm{d}x\,\psi_p(x). \tag{2.60}$$

Note that one can also include in (2.60) irrelevant operators ψ_A with $y_A < 0$. Their contributions will be relatively suppressed as ϵ^{-y_A} but one may worry that in the perturbation

[7] It should be clear from our analysis that the different technical assumptions, namely that $y_s \neq y_\phi$ for any boundary field ψ_s, and that we only have a single bulk field, are not crucial for the argument.

expansion they will lead to contributions more singular than this suppression factor. We will see that, although there is no need to introduce irrelevant operators at the leading order, they are sometimes necessary to be taken into account at higher orders in perturbation theory.

We will confine ourselves throughout this subsection to the case of a single marginal bulk field $\phi(x, y)$ $(\Delta = 2)$. In this case the terms in the perturbation expansion we are interested in are

$$
\begin{aligned}
\mathrm{e}^{\delta S} &= 1 + \lambda \iint \mathrm{d}x\, \mathrm{d}y\, \phi(x, y)\theta\left(y - \frac{\epsilon}{2}\right) + \sum_p \epsilon^{-y_p}\mu^p \int \mathrm{d}x\, \psi_p(x) \\
&\quad + \sum_p \epsilon^{-y_p}\mu^p\lambda \iint \mathrm{d}x\, \mathrm{d}y\, \theta\left(y - \frac{\epsilon}{2}\right) \int \mathrm{d}x'\, \phi(x, y)\psi_p(x')\theta(L - |x - x'|)\theta(L - y) \\
&\quad + \sum_{pq} \epsilon^{-y_p - y_q}\mu^p\mu^q \iint \mathrm{d}x_1\, \mathrm{d}x_2\, \psi_p(x_1)\psi_q(x_2)\theta(x_2 - x_1 - \epsilon)\theta(L - |x_1 - x_2|) + \cdots .
\end{aligned}
\tag{2.61}
$$

The cut-off variation $\epsilon\partial_\epsilon\, \mathrm{e}^{\delta S}$ can be computed assuming that the coupling constants depend on the cut-off via the couplings themselves according to

$$\epsilon\partial_\epsilon\mu^p = \beta^p(\mu^q, \lambda), \tag{2.62}$$

where β^p are the beta functions (2.7). In the OPE scheme we now vary $\mathrm{e}^{\delta S}$ with respect to ϵ, i.e. we compute $\epsilon\partial_\epsilon \mathrm{e}^{\delta S}$, and demand that the variation vanishes at the leading order in ϵ. This reflects the main principle of the Wilsonian renormalization group approach, namely that the renormalized quantities must be independent of the UV scale. The resulting equations fix order by order the coefficients of the beta functions. The linear terms in the beta functions are always scheme independent with the coefficients given by the anomalous dimensions. It is easy to check that at the linear order in μ^p the equation

$$\epsilon\partial_\epsilon\, \mathrm{e}^{\delta S} \underset{\epsilon\to 0}{\sim} 0 \tag{2.63}$$

is satisfied automatically. The equation arising at the linear order in λ fixes

$$\mathcal{B}^p_\phi = \tfrac{1}{2} B_\phi{}^p, \tag{2.64}$$

where $B_\phi{}^p$ are the bulk-to-boundary OPE coefficients (2.4) (see [8]). At the quadratic order in the boundary couplings one obtains the well-known expression

$$\mathcal{D}^p_{rs} = D_{rs}{}^p, \tag{2.65}$$

where $D_{rs}{}^p$ are the boundary OPE coefficients (2.2). Finally, the equation at order $\lambda\mu^q$ is

$$
\begin{aligned}
0 \underset{\epsilon\to 0}{\sim} \lambda\mu^q \Bigg[&-\frac{\epsilon^{1-y_q}}{2} \iint \mathrm{d}x\, \mathrm{d}x'\, \phi\left(x, \frac{\epsilon}{2}\right)\psi_q(x')\theta(L - |x - x'|) \\
&+ \sum_p \frac{\epsilon^{-y_p - y_q}}{2} B_\phi{}^p \iint \mathrm{d}x_1\, \mathrm{d}x_2\, \psi_q(x_1)\psi_p(x_2)\theta(|x_1 - x_2| - \epsilon)\theta(L - |x_1 - x_2|) \\
&+ \sum_p \epsilon^{-y_p}\mathcal{E}^p_{\phi q} \int \mathrm{d}x\, \psi^p(x) \Bigg].
\end{aligned}
\tag{2.66}
$$

The first line in the above expression came from applying $\epsilon\partial_\epsilon$ to the cut-off function $\theta\left(y - \frac{\epsilon}{2}\right)$ while the second line came from the lower-order term (2.64). The coefficients $\mathcal{E}^p_{\phi q}$ are formally obtained by taking correlation functions with the operator ψ_p inserted at infinity:

$$\mathcal{E}^p_{\phi q} = \left(\mathcal{E}^p_{\phi q}\right)_{\rm OPE} \equiv \frac{1}{2}\lim_{\epsilon\to 0}\epsilon^{y_p-y_q}\left[\epsilon\int_{-L}^{L}{\rm d}x\left\langle\phi\left(x,\frac{\epsilon}{2}\right)\psi_q(0)\psi_p(\infty)\right\rangle \right.$$
$$\left. -\sum_r \epsilon^{-y_r}{B_\phi}^r\int_{-L}^{L}{\rm d}x\,\langle\psi_q(0)\psi_r(x)\psi_p(\infty)\rangle\theta(|x|-\epsilon)\right]. \tag{2.67}$$

The above expression is formal because the limit may not exist. The integrals in the second line in (2.67) can be evaluated explicitly:

$$\frac{1}{2}\sum_r \epsilon^{y_p-y_q-y_r}{B_\phi}^r\int_{-L}^{L}{\rm d}x\,\langle\psi_q(0)\psi_r(x)\psi_p(\infty)\rangle\theta(|x|-\epsilon)$$
$$= \sum_r {B_\phi}^r\frac{{D_{(qr)}}^p}{(y_q-y_p+y_r)}\left[\left(\frac{L}{\epsilon}\right)^{y_q-y_p+y_r}-1\right] \equiv H(\epsilon/L). \tag{2.68}$$

To study the convergence we rewrite expression (2.67) via conformal blocks using (2.38)

$$\left(\mathcal{E}^p_{\phi q}\right)_{\rm OPE} = \lim_{\epsilon\to 0}\left[\frac{\rm i}{2}\int_{C'(\epsilon/L)}{\rm d}\eta\,\eta^{\delta-y_q+y_p-2}(1-\eta)^{-\delta/2}(\eta-1)^{(y_p-y_q)/2}Y(\eta)-H(\epsilon/L)\right], \tag{2.69}$$

where

$$C'(\epsilon/R) = \{\eta = 1-{\rm e}^{-2{\rm i}\vartheta},\,\vartheta'_*\leqslant\vartheta\leqslant\pi-\vartheta'_*\},\qquad \vartheta'_* = \frac{1}{2}\ln\left(\frac{1-{\rm i}\epsilon/2L}{1+{\rm i}\epsilon/2L}\right) \tag{2.70}$$

is a segment of a unit circle around $\eta = 1$ oriented clockwise. If we now evaluate $\tilde{\mathcal{E}}^p_{\phi p}$ of (2.16) in the OPE scheme, using (2.69) for $p = q$, as well as (2.65) and (2.64), we obtain the same expression in terms of conformal blocks as given in (2.55) and (2.59). Thus $\tilde{\mathcal{E}}^p_{\phi p}$ is indeed scheme independent, as we have argued before. In terms of correlation functions, it can now be written as

$$\tilde{\mathcal{E}}^p_{\phi p} = \lim_{\epsilon\to 0}\left[\frac{\epsilon}{2}\int_{-L}^{L}{\rm d}x\left\langle\phi\left(x,\frac{\epsilon}{2}\right)\psi_p(0)\psi_p(\infty)\right\rangle - \sum_r{D_{(pr)}}^p\frac{{B_\phi}^r}{y_r}\left(\frac{L}{\epsilon}\right)^{y_r}\right]. \tag{2.71}$$

Writing $\delta = \frac{\epsilon}{2L}$ and using the variable $\eta = 1-{\rm e}^{-2{\rm i}\vartheta}$ in the integral (2.69), we can also obtain another, perhaps more elegant expression for $\tilde{\mathcal{E}}^p_{\phi p}$:

$$\tilde{\mathcal{E}}^p_{\phi p} = \lim_{\delta\to 0}\left[\int_{\delta}^{\pi-\delta}{\rm d}\vartheta\,\langle\phi({\rm e}^{{\rm i}\vartheta})\psi_p(0)\psi_p(\infty)\rangle - \sum_r{D_{(pr)}}^p\frac{{B_\phi}^r}{y_r}\left(\frac{1}{2\delta}\right)^{y_r}\right], \tag{2.72}$$

where the bulk field insertion runs over a semicircle of radius 1 around the boundary insertion $\psi_p(0)$. Note that there is nothing special about the radius being 1, since

$$\langle\phi({\rm e}^{{\rm i}\vartheta})\psi_p(0)\psi_p(\infty)\rangle = \rho^2\langle\phi(\rho\,{\rm e}^{{\rm i}\vartheta})\psi_p(0)\psi_p(\infty)\rangle \tag{2.73}$$

for any $\rho > 0$.

Let us now come back to expression (2.69) for $p \neq q$. Substituting the asymptotic expansion (2.47) into (2.69) we find that although the most dangerous divergences, associated with the leading contributions of relevant primaries in (2.47), cancel out, there may be divergences coming from terms in (2.47) associated with irrelevant fields. More precisely, there are additional divergences in $\left(\mathcal{E}^p_{\phi q}\right)_{\rm OPE}$ from the region near $\eta\sim 0$ whenever there is an irrelevant primary $\phi_A(z)$ in the theory such that

$${B_\phi}^A \neq 0 \qquad \text{and} \qquad \left\{{D_{qA}}^p\neq 0 \text{ or } {D_{Aq}}^p\neq 0\right\} \qquad \text{and} \qquad y_A+y_q > y_p, \tag{2.74}$$

or whenever we have a relevant primary $\psi_r(z)$ such that

$$B_\phi{}^r \neq 0 \qquad \text{and} \qquad \left\{D_{qr}{}^p \neq 0 \text{ or } D_{rq}{}^p \neq 0\right\} \qquad \text{and} \qquad y_r + y_q > y_p + 1. \tag{2.75}$$

In the last case, the irrelevant field causing the divergence is a descendant of the primary ψ_r. From the point of view of the minimal subtraction scheme of the previous subsection, this problem does not arise since the conditions (2.74) and (2.75) imply that there are no divergences from the region where the bulk field approaches the boundary insertion. In fact, the additional $\epsilon \to 0$ divergences in the OPE scheme come directly from the extra divergent factors of ϵ included in the action (2.60).

The situation can be mended if we include in the original perturbed action (2.60) also irrelevant fields, and introduce their beta functions by requiring that $\epsilon\partial_\epsilon\, \mathrm{e}^{\delta S} \sim 0$ at the subleading orders in ϵ. Then the function $H(\epsilon/L)$ is modified accordingly to include more divergent terms that cancel out the divergences coming from the integral in (2.69). Although this resolution looks quite natural from the Wilsonian point of view, the whole scheme becomes quite unwieldy for practical applications whenever (2.74) or (2.75) happens. Note, however, that these extra divergences do *not* appear for universal quantities like $\tilde{\mathcal{E}}^p_{\phi p}$. Thus, as long as we are only interested in these quantities we can (and will) use the technically simpler OPE scheme. In particular, we will use this method to compute analogous quantities for pure bulk and pure boundary perturbations in section 4.

It is worth noting that the complications related to (2.74) and (2.75) arise only in the presence of several running coupling constants. Although beta function coefficients were studied for some models to a very large order, see e.g. [16], such computations typically involved only a single coupling constant.

2.4. Perturbations by boundary changing operators

Up to now we have assumed that there is a single (fundamental) boundary condition. The whole analysis can easily be generalized to the situation where we have superpositions of boundary conditions; in that case the set of boundary operators includes also boundary changing operators $\psi_p^{ab}(x)$, where the two boundary conditions are labelled by a and b, with a being the boundary condition to the left of x and b to the right. Local excitations of the pure boundary a are denoted ψ_p^{aa}. The study of renormalization group flows involving such operators was initiated in [24].

The OPEs of a bulk field approaching the boundary with label a, and that of two boundary fields have the form

$$\phi_i(x+\mathrm{i}y, x-\mathrm{i}y) = \sum_r {}^aB_i{}^r\,(2y)^{h_r-\Delta_i}\,\psi_r^{aa} + \cdots, \tag{2.76}$$

$$\psi_p^{ab}(x)\psi_q^{bc}(y) = \sum_r D_{pq}^{(abc)r}\,(y-x)^{h_r-h_p-h_q}\,\psi_r(y) + \cdots \quad (y > x). \tag{2.77}$$

The only difference to the previous analysis is that there are now various superselection rules that demand, for example, that products of boundary operators can only be non-zero if the intermediate boundary conditions match, or that the boundary fields that appear in the bulk-to-boundary OPE are always boundary-preserving fields. Taking this into account, the boundary beta functions then have the following general form:

$$\beta_p^{ab} = y_p^{ab}\mu^{p(ab)} + {}^a\mathcal{B}_\phi^p\delta^{ab}\lambda + \sum_{c;rs}\mathcal{D}_{rs}^{p(acb)}\mu^{r(ac)}\mu^{s(cb)} + \sum_r \mathcal{E}_{\phi r}^{p(ab)}\lambda\mu^{r(ab)} + \cdots, \tag{2.78}$$

where $y_p^{ab} = 1 - h_p^{ab}$ are anomalous dimensions, and $\mu^{r(ab)}$ are the coupling constants of the operators $\psi_r^{ab}(x)$. The expression for the dimension shift (2.16) generalizes as

$$\tilde{\mathcal{E}}_{\phi p}^{p(ab)} = \mathcal{E}_{\phi p}^{p(ab)} - \sum_r \mathcal{D}_{rp}^{p(aab)} \frac{{}^a\mathcal{B}_\phi^r}{y_r^{aa}} - \sum_r \mathcal{D}_{pr}^{p(abb)} \frac{{}^b\mathcal{B}_\phi^r}{y_r^{bb}}. \tag{2.79}$$

Similarly, the main results of the previous subsections (2.71) and (2.72) now become

$$\tilde{\mathcal{E}}_{\phi p}^{p(ab)} = \lim_{\epsilon\to 0} \frac{1}{2}\left[\epsilon \int_{-L}^{L} \mathrm{d}x \left\langle \phi\left(x, \frac{\epsilon}{2}\right) \psi_p^{ab}(0) \psi_p^{ba}(\infty)\right\rangle \right.$$
$$\left. - \sum_r D_{rp}^{(aab)p} \frac{{}^aB_\phi{}^r}{y_r^{aa}} \left(\frac{L}{\epsilon}\right)^{y_r^{aa}} - \sum_r D_{pr}^{(abb)p} \frac{{}^bB_\phi{}^r}{y_r^{bb}} \left(\frac{L}{\epsilon}\right)^{y_r^{bb}} \right], \tag{2.80}$$

$$\tilde{\mathcal{E}}_{\phi p}^{p(ab)} = \lim_{\delta\to 0}\left[\int_{\delta}^{\pi-\delta} \mathrm{d}\vartheta \left\langle \phi(\mathrm{e}^{\mathrm{i}\vartheta}) \psi_p^{ab}(0) \psi_p^{ba}(\infty)\right\rangle \right.$$
$$\left. - \sum_r D_{rp}^{(aab)p} \frac{{}^aB_\phi{}^r}{2y_r^{aa}} \left(\frac{1}{2\delta}\right)^{y_r^{aa}} - \sum_r D_{pr}^{(abb)p} \frac{{}^bB_\phi{}^r}{2y_r^{bb}} \left(\frac{1}{2\delta}\right)^{y_r^{bb}} \right]. \tag{2.81}$$

3. Some explicit examples

Up to now our analysis has been very general. In this section, we want to illustrate these general results with two simple examples.

3.1. A single Neumann brane

The simplest example is the case of a single Neumann brane on a circle of radius R. The action of this theory is simply[8]

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z\, \partial X\, \bar{\partial} X. \tag{3.1}$$

The bulk field that corresponds to changing the radius R is $\phi(z, \bar{z}) = 2\partial X(z) \bar{\partial} X(\bar{z})$, which is an exactly marginal operator in the bulk. More specifically, we shall consider the perturbation

$$\delta S = 2\lambda \int \mathrm{d}x\, \mathrm{d}y\, \partial X(w) \bar{\partial} X(\bar{w}) \tag{3.2}$$

that changes the radius R as $R^\lambda = R\,\mathrm{e}^{-\pi\lambda}$ so that to the first order we have $\delta R = -\pi R\lambda$.

On the Neumann brane we have open string momentum states corresponding to the vertex operators $\psi = \mathrm{e}^{\mathrm{i}kX}$, whose conformal dimension is $h = k^2$ with $k = \frac{n}{R}$ and $n \in \mathbb{Z}$. We want to study how the conformal dimension of these operators changes as we change the radius. Thus we need to calculate[9]

$$\mathcal{E} = \lim_{\delta\to 0}\left[2\int_{\delta}^{\pi-\delta} \mathrm{d}\vartheta \,\langle \mathrm{e}^{-\mathrm{i}kX}(\infty) \partial X(\mathrm{e}^{\mathrm{i}\vartheta}) \bar{\partial} X(\mathrm{e}^{-\mathrm{i}\vartheta})\, \mathrm{e}^{\mathrm{i}kX}(0)\rangle - \sum_r D_{r\psi}{}^{\psi} \frac{B_\phi{}^r}{y_r} \left(\frac{1}{2\delta}\right)^{y_r}\right]. \tag{3.3}$$

On the Neumann boundary we have $\partial X = \bar{\partial} X$, and the correlation function equals

$$2\langle \mathrm{e}^{-\mathrm{i}kX}(\infty) \partial X(\mathrm{e}^{\mathrm{i}\vartheta}) \bar{\partial} X(\mathrm{e}^{-\mathrm{i}\vartheta})\, \mathrm{e}^{\mathrm{i}kX}(0)\rangle = -2k^2 - \frac{1}{(z - \bar{z})^2}. \tag{3.4}$$

[8] Throughout this section we set $\alpha' = 1$.

[9] Note that ψ is not a self-conjugate field, and we therefore have to insert the conjugate field at infinity.

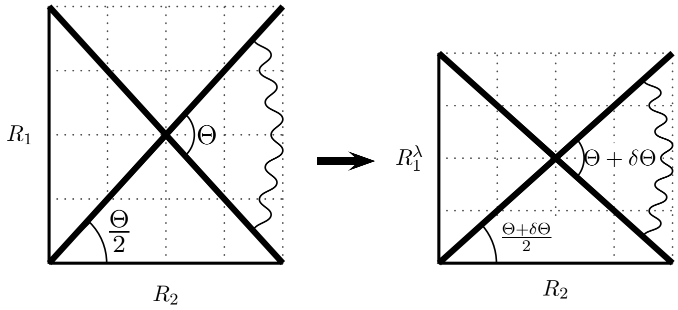

Figure 1. Radius perturbation on the torus: changing the radius of R_1 modifies the relative angle between the two branes, and hence the conformal dimension of the corresponding boundary changing field.

Thus the integral is simply

$$2\int_{\delta}^{\pi-\delta} \mathrm{d}\vartheta \, \langle \mathrm{e}^{-\mathrm{i}kX}(\infty)\partial X(\mathrm{e}^{\mathrm{i}\vartheta})\bar{\partial}X(\mathrm{e}^{-\mathrm{i}\vartheta})\,\mathrm{e}^{\mathrm{i}kX}(0)\rangle = -2k^2\pi - \frac{1}{4}\cot\vartheta\Big|_{\vartheta=\delta}^{\vartheta=\pi-\delta}$$
$$= -2k^2\pi + \frac{1}{2\delta} + \mathcal{O}(\delta). \tag{3.5}$$

The term that is singular in δ is subtracted by the last term in (3.3). In fact, the only relevant or marginal boundary field that is switched on is the identity field with $y_0 = 1$ and $D_{\mathbf{1}\psi}{}^{\psi} = 1$, and the corresponding bulk-to-boundary OPE coefficient is $B_\phi{}^{\mathbf{1}} = 1$ since

$$2\langle\partial X(z)\bar{\partial}X(\bar{z})\rangle = -\frac{1}{(z-\bar{z})^2} = \frac{1}{4y^2}. \tag{3.6}$$

Thus we find that $\mathcal{E} = -2k^2\pi$ in this example, which implies that

$$\delta h = 2k^2\pi\lambda. \tag{3.7}$$

This then agrees with the geometrical expectation since for $h = k^2$ with $k = \frac{n}{R}$ we have

$$\delta h = -2k^2\frac{\delta R}{R} = 2k^2\pi\lambda. \tag{3.8}$$

3.2. Branes at angles

A somewhat more interesting example is the configuration of two D1-branes that stretch diagonally across a 2-torus, crossing each other at an angle (see figure 1). This brane configuration is obviously unstable since the relative open string between the two D1-branes is tachyonic but this will not be important in the following. (One can imagine that this is only part of a more complicated background involving additional directions, and that the boundary conditions of these D-branes in the other directions are chosen so that the relative open string is not tachyonic.)

For simplicity, consider the situation where the T^2 torus is orthogonal with radii R_1 and R_2. The bulk operator that changes either radius is an exactly marginal bulk operator, but it does have an important impact on the boundary theory since the ratio of the two radii determines the conformal dimension of the lowest string excitation between the two D1-branes. In the following (section 3.2.2), we shall calculate the change in conformal dimension using the RG formalism we have developed above. As we shall see, this will reproduce the standard formula for the conformal dimension of boundary fields on branes at angles that will be reviewed in section 3.2.1.

3.2.1. The geometrical analysis. The torus theory is described by the action

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z\, \partial X^\mu \bar{\partial} X_\mu. \tag{3.9}$$

We shall now consider changing the radius R_1 by means of the perturbation

$$\delta S = 2\lambda \int \mathrm{d}x\, \mathrm{d}y\, \partial X^1(w)\bar{\partial}X^1(\bar{w}), \tag{3.10}$$

which to first order gives $\delta R_1 = -\pi R_1 \lambda$.

The two D-branes stretch diagonally across the torus; their angle relative to the x^2-axis will be denoted by $\pm\Theta/2$, where Θ satisfies

$$\tan\frac{\Theta}{2} = \frac{R_1}{R_2}. \tag{3.11}$$

The open string that stretches between the two branes satisfies the + boundary condition

$$\begin{aligned} \partial X^1(z) &= -\cos\Theta\bar{\partial}X^1(\bar{z}) + \sin\Theta\bar{\partial}X^2(\bar{z}), \\ \partial X^2(z) &= \sin\Theta\bar{\partial}X^1(\bar{z}) + \cos\Theta\bar{\partial}X^2(\bar{z}) \end{aligned} \tag{3.12}$$

at one end (say for $z = \bar{z}$ on the positive real axis), and the − boundary condition

$$\begin{aligned} \partial X^1(z) &= -\cos\Theta\bar{\partial}X^1(\bar{z}) - \sin\Theta\bar{\partial}X^2(\bar{z}), \\ \partial X^2(z) &= -\sin\Theta\bar{\partial}X^1(\bar{z}) + \cos\Theta\bar{\partial}X^2(\bar{z}) \end{aligned} \tag{3.13}$$

at the other (say for $z = \bar{z}$ on the negative real axis). By going to complex variables, i.e. by writing $Z^+ = \frac{1}{\sqrt{2}}(X^1 + \mathrm{i}X^2)$, $Z^- = \frac{1}{\sqrt{2}}(X^1 - \mathrm{i}X^2)$, we can write the open string fields as

$$Z^+(z,\bar{z}) = \mathrm{i}\sqrt{\frac{1}{2}}\sum_{m\in\mathbb{Z}}\left(\frac{a^+_{m-\nu}}{(m-\nu)z^{m-\nu}} - \mathrm{e}^{-\mathrm{i}\Theta}\frac{a^-_{m+\nu}}{(m+\nu)\bar{z}^{m+\nu}}\right) \tag{3.14}$$

$$Z^-(z,\bar{z}) = \mathrm{i}\sqrt{\frac{1}{2}}\sum_{m\in\mathbb{Z}}\left(\frac{a^-_{m+\nu}}{(m+\nu)z^{m+\nu}} - \mathrm{e}^{\mathrm{i}\Theta}\frac{a^+_{m-\nu}}{(m-\nu)\bar{z}^{m-\nu}}\right), \tag{3.15}$$

where $\nu = \Theta/\pi \in [0, 1)$. The modes $a^\pm_{m\mp\nu}$ satisfy the canonical commutation relations

$$\left[a^+_{m-\nu}, a^-_{n+\nu}\right] = (m-\nu)\delta_{m,-n}, \tag{3.16}$$

and the Virasoro generators can be expressed in terms of them as

$$L_m = \sum_{k\in\mathbb{Z}} : a^+_{m-k-\nu}a^-_{k+\nu} : + \frac{1}{2}\nu(1-\nu)\delta_{m,0}. \tag{3.17}$$

The conformal dimension of the lowest boundary changing operator ψ^{-+} is thus

$$h^{-+}_\psi = \tfrac{1}{2}\nu(1-\nu). \tag{3.18}$$

According to the analysis of [8, 25], the two D-branes will respond to the radius changing bulk perturbation (3.10) by simply adjusting themselves infinitesimally, so that they continue to stretch diagonally across. To first order in λ, the angle Θ thus changes via (3.11) as

$$\delta\Theta = -\pi\sin\Theta\lambda. \tag{3.19}$$

With $\pi\nu = \Theta$ this implies that the conformal dimension of the lowest boundary changing operator changes as

$$\delta h^{-+}_\psi = \tfrac{1}{2}(2\nu - 1)\sin\Theta\lambda + \mathcal{O}(\lambda^2). \tag{3.20}$$

This is the result we now want to reproduce using the RG approach explained above.

3.2.2. The RG approach. Formula (2.81) applied to the situation at hand reads

$$\tilde{\mathcal{E}}^{\psi(-+)}_{\phi\psi} = \lim_{\delta\to 0}\Bigg[2\int_{\delta}^{\pi-\delta} \mathrm{d}\vartheta\, \langle \partial X^1(\mathrm{e}^{\mathrm{i}\vartheta})\bar\partial X^1(\mathrm{e}^{-\mathrm{i}\vartheta})\psi^{-+}(0)\psi^{+-}(\infty)\rangle \\ - \sum_r D^{(--+)\psi}_{r\psi}\frac{{}^{-}B_{\phi}{}^{r}}{2y_r^{--}}\left(\frac{1}{2\delta}\right)^{y_r^{--}} - \sum_r D^{(-++)\psi}_{\psi r}\frac{{}^{+}B_{\phi}{}^{r}}{2y_r^{++}}\left(\frac{1}{2\delta}\right)^{y_r^{++}}\Bigg], \tag{3.21}$$

where the index r runs over all relevant boundary operators in the respective sectors. Since the model at hand is Gaussian the only relevant operator induced on the boundary by $\phi(z,\bar z)$ is the identity operator in the respective + or − sector. The corresponding bulk-to-boundary OPE coefficients can be read off from the expectation values

$$2\langle \partial X^1(z)\bar\partial X^1(\bar z)\rangle_{\pm} = \frac{{}^{\pm}B_{\phi}{}^{\mathbf{1}}}{4y^2} = -\frac{\cos\Theta}{4y^2} \tag{3.22}$$

which can be computed using the mode expansions (3.14) and (3.15) for $\nu = 0$. (The string fields in the presence of a single boundary are of the same form as (3.14) and (3.15) but with $\nu = 0$.) Thus

$${}^{+}B_{\phi}{}^{\mathbf{1}} = {}^{-}B_{\phi}{}^{\mathbf{1}} = -\cos\Theta. \tag{3.23}$$

The three-point correlator in (3.21) is given by the one-point function of the radius changing operator in the presence of the boundary conditions (3.12) and (3.13). A straightforward computation yields

$$2\langle \partial X^1(z)\bar\partial X^1(\bar z)\psi^{-+}(0)\psi^{+-}(\infty)\rangle = 2\langle \partial X^1(z)\bar\partial X^1(\bar z)\rangle_{\Theta} \\ = \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\Theta}\frac{z^{\nu}}{\bar z^{\nu}}\frac{z(1-\nu)+\bar z\nu}{z(z-\bar z)^2} + \text{c.c.} \tag{3.24}$$

Viewed as a function on $\mathbb{C}$ rather than on $\mathbb{H}^{+}$, the correlator has a logarithmic branch cut along the negative real axis. The integral at hand can be easily evaluated:

$$2\int_{\delta}^{\pi-\delta}\mathrm{d}\vartheta\,\langle \partial X^1(z)\bar\partial X^1(\bar z)\rangle_{\Theta} = -\int_{\delta}^{\pi-\delta}\mathrm{d}\vartheta\,\frac{\mathrm{e}^{-\mathrm{i}\Theta+2\mathrm{i}\nu\vartheta}}{8\sin^2\vartheta}(1-\nu+\nu\,\mathrm{e}^{-2\mathrm{i}\vartheta}) + \text{c.c.} \\ = \frac{\cos(-\Theta+\vartheta(2\nu-1))}{4\sin\vartheta}\Bigg|_{\vartheta=\delta}^{\vartheta=\pi-\delta} = -\frac{\cos(-\Theta+\delta(2\nu-1))}{2\sin\delta}, \tag{3.25}$$

where in the last step we used $\Theta = \pi\nu$. Substituting (3.25) and (3.23) into (3.21) we obtain

$$\tilde{\mathcal{E}}^{\psi(-+)}_{\phi\psi} = \lim_{\delta\to 0}\left[-\frac{\cos(-\Theta+\delta(2\nu-1))}{2\sin\delta} + \frac{\cos\Theta}{2\delta}\right] = -\frac{1}{2}(2\nu-1)\sin\Theta. \tag{3.26}$$

Noting that

$$\delta h_{\psi}^{-+} = -\delta y_{\psi}^{-+} = -\lambda\tilde{\mathcal{E}}^{\psi(-+)}_{\phi\psi} \tag{3.27}$$

we finally get the same result as in (3.20).

4. Third-order coefficients in the pure bulk or boundary case

In section 2, we explained how to calculate higher-order coefficients in a theory with bulk and boundary perturbations. Actually, the techniques used there can also be easily generalized to the pure bulk or pure boundary case; this will be sketched in the following. We begin with a discussion about which coefficients of the third-order terms in the beta functions contain universal quantities, paralleling the discussion in section 2.1. In section 4.2, we then describe how to obtain useful formulae for these coefficients.

4.1. Universal quantities

In this subsection we shall only consider the pure bulk theory; the discussion for the pure boundary case is very similar. Consider a conformal field theory perturbed by

$$\delta S = \sum_i \lambda^i l^{-y_i} \int \mathrm{d}^2 z \, \phi_i(z,\bar{z}), \tag{4.1}$$

where as before $y_l = 2 - \Delta_l$ are the anomalous dimensions, l is a renormalization length scale, and λ^i are the dimensionless coupling constants. The beta functions have the general form

$$\beta^l = y_l \lambda^l + \sum_{ij} \mathcal{C}^l_{ij} \lambda^i \lambda^j + \sum_{ijk} \mathcal{F}^l_{ijk} \lambda^i \lambda^j \lambda^k + \mathcal{O}(\lambda^4). \tag{4.2}$$

We take the constants $\mathcal{C}^l_{ij}$ and $\mathcal{F}^l_{ijk}$ to be totally symmetric in i, j and i, j, k, respectively. Under a general change of the scheme the coupling constants are redefined as

$$\tilde{\lambda}^l := \lambda^l + \sum_{ij} c^l_{ij} \lambda^i \lambda^j + \sum_{ijk} f^l_{ijk} \lambda^i \lambda^j \lambda^k + \mathcal{O}(\lambda^4), \tag{4.3}$$

where the c^l_{ij} are, without loss of generality, symmetric in i and j. The beta functions in the new scheme are

$$\begin{aligned}\tilde{\beta}^l = y_l \tilde{\lambda}^l &+ \sum_{ij} \tilde{\lambda}^i \tilde{\lambda}^j \left(\mathcal{C}^l_{ij} + c^l_{ij}(y_i + y_j - y_l)\right) + \sum_{ijk} \tilde{\lambda}^i \tilde{\lambda}^j \tilde{\lambda}^k \Bigg[\mathcal{F}^l_{ijk} + f^l_{ijk}(y_i + y_j + y_k - y_l) \\ &+ \frac{1}{3} \sum_m \sum_{\mathrm{perm}(i,j,k)} \left(c^l_{mi}\mathcal{C}^m_{jk} - \mathcal{C}^l_{mi} c^m_{jk} - c^l_{mi} c^m_{jk}(y_m + y_i - y_l)\right)\Bigg] + \mathcal{O}(\tilde{\lambda}^4).\end{aligned} \tag{4.4}$$

We observe that the second-order coefficients $\mathcal{C}^l_{ij}$ do not change under this transformation if and only if the second-order resonance condition $y_i + y_j = y_l$ is satisfied. As for the coefficients $\mathcal{F}^l_{ijk}$ at the cubic powers of the couplings, it can be seen from (4.4) that the basic requirement for $\mathcal{F}^l_{ijk}$ to be universal is that it satisfies the resonance condition $y_i + y_j + y_k = y_l$. However, even if the resonance condition is satisfied, the third line in (4.4) shows that the corresponding coefficient may not be invariant under general scheme changes because of the lower-order coefficients $\mathcal{C}^l_{ij}$. The resulting transformations of the resonant coefficients are parametrized by the tensors c^k_{ij}. For an n-dimensional coupling space the dimension of the space of coefficients $\mathcal{F}^l_{ijk}$ is $\frac{n^2(n+1)(n+2)}{6}$ while that of the coefficients c^k_{ij} is $\frac{n^2(n+1)}{2}$. Depending on how many coefficients are resonant, there may be some functions defined on these resonant coefficients which are invariant and thus give universal quantities. For example, if all couplings are marginal ($y_i = 0$ for all i) then generically there must be a subspace of scheme-independent coefficients of dimension $\frac{n^2(n+1)(n-1)}{6}$.

While in general it is hard to write out explicit expressions for universal quantities in terms of $\mathcal{F}^l_{ijk}$ and $\mathcal{C}^l_{ij}$ we can do so in the absence of second-order resonances because we can then use a special scheme in which all $\mathcal{C}^l_{ij}$ vanish. Given a cubic resonance $y_i + y_j + y_k = y_l$, the values of the cubic coefficients $\tilde{\mathcal{F}}^l_{ijk}$ in that scheme are universal and can be expressed via the coefficients in an arbitrary scheme as

$$\tilde{\mathcal{F}}^l_{ijk} = \mathcal{F}^l_{ijk} + \frac{1}{3} \sum_{\mathrm{perm}(i,j,k)} \sum_m \frac{\mathcal{C}^m_{ij}\mathcal{C}^l_{mk}}{y_l - y_k - y_m}. \tag{4.5}$$

The scheme independence of (4.5) can be checked directly using (4.3) and (4.4).

We can also consider a situation analogous to that considered in section 2.1 when the universal quantity gives dimension shifts under a truly marginal deformation. Let λ be a coupling constant corresponding to an exactly marginal operator $\phi(z, \bar{z})$. The beta functions for the other operators $\phi_l(z, \bar{z})$ have the form

$$\beta^l = \sum_k \left(y_l \delta^l_k + \lambda \mathcal{C}^l_{\phi k} + \lambda^2 \mathcal{F}^l_{\phi\phi k}\right)\lambda^k + \sum_{ij} \left(\mathcal{C}^l_{ij} + \lambda \mathcal{F}^l_{\phi ij}\right)\lambda^i \lambda^j + \sum_{ijk} \mathcal{F}^l_{ijk} \lambda^i \lambda^j \lambda^k + \lambda^2 \mathcal{C}^l_{\phi\phi} + \lambda^3 \mathcal{F}^l_{\phi\phi\phi} + \cdots. \tag{4.6}$$

We can make the beta functions β^i homogeneous in λ^i up to cubic order by a coupling constant redefinition

$$\tilde{\lambda}^i = \lambda^i + \frac{\mathcal{C}^i_{\phi\phi}}{y_i}\lambda^2 + \frac{\mathcal{F}^i_{\phi\phi\phi}}{y_i}\lambda^3. \tag{4.7}$$

This redefinition is possible because λ is truly marginal. The anomalous dimensions of the operators ϕ_i are then given by the eigenvalues of the matrix

$$D^j_i(\lambda) \equiv \left(\frac{\partial \tilde{\beta}^j}{\partial \tilde{\lambda}^i}\right)_{\tilde{\lambda}^k = 0} \tag{4.8}$$

and have the form

$$y_i[\lambda] = y_i + \lambda \delta^{(1)}_i + \lambda^2 \delta^{(2)}_i + \cdots. \tag{4.9}$$

A straightforward computation yields

$$\delta^{(1)}_i = \mathcal{C}^i_{\phi i} \tag{4.10}$$

and

$$\delta^{(2)}_i = \mathcal{F}^i_{\phi\phi i} - 2\sum_k \frac{\mathcal{C}^k_{\phi\phi}\mathcal{C}^i_{ik}}{y_k} + \sum_{k \neq i} \frac{\mathcal{C}^k_{\phi i}\mathcal{C}^i_{\phi k}}{y_i - y_k}. \tag{4.11}$$

The coefficients $\mathcal{C}^i_{\phi i}$ are resonant and thus universal. One can also check that (4.11) is invariant under an arbitrary coupling constants redefinition of the form

$$\tilde{\lambda}^l = f^l(\lambda) + \sum_k f^l_k(\lambda)\lambda^k + \sum_{ik} f^l_{ik}(\lambda)\lambda^i\lambda^k + \sum_{ijk} f^l_{ijk}\lambda^i\lambda^j\lambda^k, \tag{4.12}$$

where f^l, f^l_k, f^l_{ik} are polynomial functions of λ.

Finally, let us mention the well-known fact that if there is a single running coupling constant whose UV dimension is marginal, then both the quadratic and cubic terms in its beta function are universal.

4.2. Computation of coefficients

Now that we have understood which coefficients are universal, we can ask how they can be calculated explicitly. As in the bulk–boundary case discussed in section 2, we can either use a minimal subtraction scheme (see section 2.2) or the OPE scheme of section 2.3. As before, the minimal subtraction scheme is conceptually clearer since one does not need to introduce beta functions for irrelevant fields. However, the calculation is somewhat unwieldy in this scheme, since one has to isolate the divergences in the UV cut-off ϵ for finite IR cut-off L.

In the following, we shall only consider universal quantities for which the calculation in either scheme must give the same answer. Since the OPE scheme is technically simpler, we shall use it to determine explicit expressions for these coefficients. We have also checked that our result agrees with what would have been obtained in the minimal subtraction scheme (as must be the case). Moreover, for brevity we will focus on the quantity (4.5). It is straightforward to extend our results to the dimension shifts (4.11) and to a cubic term in a beta function of a single marginal coupling.

4.2.1. Resonant bulk coefficients. In the OPE scheme the RG equations are determined from the condition that the variation $\epsilon\partial_\epsilon\, \mathrm{e}^{\delta S}$ vanishes in the limit $\epsilon \to 0$. As before we regularize the theory by point splitting, i.e. we introduce a sharp UV cut-off ϵ. In addition we have an IR cut-off L. To cubic order in the couplings we have

$$\begin{aligned}\mathrm{e}^{\delta S} = 1 + \sum_i \lambda^i \epsilon^{-y_i} \int \mathrm{d}^2 z\, \phi_i(z,\bar z) + \frac{1}{2!}\sum_{ij} \lambda^i \lambda_j \epsilon^{-y_i - y_j} \\ \times \iint \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \theta_{12} \phi_i(z_1,\bar z_1)\phi_j(z_2,\bar z_2) + \frac{1}{3!}\sum_{ijk}\lambda^i\lambda^j\lambda^k \epsilon^{-y_i-y_j-y_k} \\ \times \iiint \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3\, \theta_{12}\theta_{23}\theta_{13}\phi_i(z_1,\bar z_1)\phi_j(z_2,\bar z_2)\phi_k(z_3,\bar z_3) + \cdots,\end{aligned}$$

where $\theta_{ij} = \theta(|z_i - z_j| - \epsilon)\theta(L - |z_i - z_j|)$. The variation $\epsilon\partial_\epsilon$ of this expression can be computed using (4.2). Setting $\epsilon\partial_\epsilon\, \mathrm{e}^{\delta S} \sim 0$ at second order in the couplings one obtains the well-known expression

$$\mathcal{C}_{ij}^m = \pi C_{ij}{}^m, \tag{4.13}$$

where $C_{ij}{}^m$ are the bulk OPE coefficients (2.2). At the cubic order we have the equation

$$\begin{aligned}0 \underset{\epsilon\to 0}{\sim} \lambda^i\lambda^j\lambda^k \sum_{\mathrm{perm}(i,j,k)} \Bigg[-\frac{1}{2}\epsilon^{-y_i-y_j-y_k+1}\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^3 z_3\, \delta_{12}^{\epsilon}\theta_{13}\theta_{23}\phi_i(z_1,\bar z_1)\phi_j(z_2,\bar z_2)\phi_k(z_3,\bar z_3) \\ + \sum_m \pi C_{ij}{}^m \epsilon^{-y_m-y_k}\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \theta_{12}\phi_m(z_1,\bar z_1)\phi_k(z_2,\bar z_2) \\ + \sum_l \mathcal{F}_{ijk}^l \epsilon^{-y_l}\int \mathrm{d}^2 z\, \phi_l(z,\bar z)\Bigg].\end{aligned} \tag{4.14}$$

As was discussed in section 2.3 the above equation in general may still have divergences. However, in the resonant case, i.e. if $y_i + y_j + y_k = y_l$ no such complications arise. Then we can write

$$\begin{aligned}\left(\mathcal{F}_{ijk}^l\right)_{\mathrm{res}} = \lim_{\epsilon\to 0}\frac{\epsilon}{12}\int \mathrm{d}^2 z_2\, \theta_{20} \sum_{\mathrm{perm}(ijk)}\Bigg[\int \mathrm{d}^2 z_1\, \theta_{10}\delta_{12}^{\epsilon}\langle\phi_i(z_1,\bar z_1)\phi_j(z_2,\bar z_2)\phi_k(0)\phi_l(\infty)\rangle \\ -2\pi\sum_m \epsilon^{y_i+y_j-y_m-1} C_{ij}{}^m\langle\phi_m(z_2,\bar z_2)\phi_k(0)\phi_l(\infty)\rangle\Bigg],\end{aligned} \tag{4.15}$$

where $\theta_{10} = \theta(|z_1| - \epsilon)\theta(L - |z_1|)$, and similarly for θ_{20}. As before, we consider spinless fields, for which we can express the four-point correlator in terms of conformal blocks,

$$\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\phi_3(z_3,\bar z_3)\phi_4(z_4,\bar z_4)\rangle = \prod_{i<j}|z_{ij}|^{2(\delta-h_i-h_j)}Y_{1234}(\eta,\bar\eta), \tag{4.16}$$

with

$$z_{ij} = z_i - z_j, \qquad \delta = \frac{1}{3}\sum_{i=1}^4 h_i, \qquad \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{4.17}$$

and

$$Y_{1234}(\eta,\bar\eta) = \begin{cases}\sum_m C_{12}{}^m C_{m3}{}^4 F_{12,34}^m(\eta)\bar F_{12,34}^m(\bar\eta) \\ \sum_m C_{32}{}^m C_{m1}{}^4 F_{32,14}^m(1-\eta)\bar F_{32,14}^m(1-\bar\eta) \\ \sum_m C_{13}{}^m C_{m2}{}^4 F_{13,24}^m(1/\eta)\bar F_{13,24}^m(1/\bar\eta).\end{cases} \tag{4.18}$$

The conformal blocks are normalized such that

$$F^{m}_{12,34}(\eta) \sim \eta^{h_m-\delta} \qquad \tilde{F}^{m}_{12,34}(\bar{\eta}) \sim \bar{\eta}^{h_m-\delta} \qquad \text{for} \quad \eta, \bar{\eta} \to 0. \tag{4.19}$$

Replacing the variables in the first part of the square brackets in (4.15) by an angular variable φ and the cross ratio η,

$$z_1 = z_2 + \epsilon\, \mathrm{e}^{\mathrm{i}\varphi}, \qquad z_2 = \epsilon\, \mathrm{e}^{\mathrm{i}\varphi} \frac{1-\eta}{\eta},$$

the angular variable can be integrated out by the integral over z_1, and one finds

$$\begin{aligned}\left(\mathcal{F}^{l}_{ijk}\right)_{\text{res}} = \lim_{\epsilon\to 0} \frac{\pi}{6} \int \mathrm{d}^2\eta \sum_{\text{perm}(i,j,k)} \theta(1-|\eta|)\theta\left(|\eta| - \frac{\epsilon}{L}\right) \\ \times \left\{\theta\left(\frac{1}{2} - \mathrm{Re}\,\eta\right)\theta\left(|\eta|^2\left(\frac{L^2}{\epsilon^2}-1\right) + 2\,\mathrm{Re}\,\eta - 1\right)|\eta|^{-y_i-y_j-2y_k-4\delta+4}\right. \\ \left. \times |1-\eta|^{2\delta+y_j+y_k-4} Y_{ij,kl}(\eta,\bar{\eta}) - \sum_m C_{ij}{}^{m} C_{mk}{}^{l} |\eta|^{y_l-y_m-y_k-2}\right\}. \end{aligned} \tag{4.20}$$

In the second term in the bracket of (4.15), we changed variables to $z_2 = \epsilon/\eta$. The function in the last line in (4.20) can be integrated explicitly, and we obtain

$$\begin{aligned}\left(\mathcal{F}^{l}_{ijk}\right)_{\text{res}} = \lim_{\epsilon\to 0} \sum_{\text{perm}(i,j,k)} \left[\frac{\pi}{6}\int \mathrm{d}^2\eta\, \theta(1-|\eta|)\theta\left(|\eta| - \frac{\epsilon}{L}\right)\theta\left(\frac{1}{2} - \mathrm{Re}\,\eta\right)|\eta|^{-y_i-y_j-2y_k-4\Delta+4}\right. \\ \left. \times |1-\eta|^{2\Delta+y_j+y_k-4} Y_{ij,kl}(\eta,\bar{\eta}) - \frac{\pi^2}{3}\sum_m \frac{C_{ij}{}^{m} C_{mk}{}^{l}}{y_l - y_k - y_m}\left(\frac{\epsilon}{L}\right)^{y_l-y_k-y_m}\right] \\ + \frac{\pi^2}{3}\sum_{\text{perm}(i,j,k)}\sum_m \frac{C_{ij}{}^{m} C_{mk}{}^{l}}{y_l - y_k - y_m}. \end{aligned} \tag{4.21}$$

The universal quantity $\tilde{\mathcal{F}}^{l}_{ijk}$ defined in (4.5) is then simply given by the first two lines of (4.21).

It can be checked using the asymptotics (4.19) and the properties of conformal blocks (4.18) that the integral (4.20) converges in the regions $\eta \sim \epsilon/L \to 0$, $|1-\eta| \sim \epsilon/L \to 0$ and $|\eta| \sim L/\epsilon \to \infty$. One can thus safely set $\epsilon = 0$ in (4.20) to obtain an integral expression

$$\begin{aligned}\left(\mathcal{F}^{l}_{ijk}\right)_{\text{res}} = \frac{\pi}{6}\int \mathrm{d}^2\eta \sum_{\text{perm}(i,j,k)} \theta(1-|\eta|)\theta\left(\frac{1}{2} - \mathrm{Re}\,\eta\right) \\ \times \left\{|\eta|^{2r+y_i+y_j-4}|1-\eta|^{2r+y_j+y_k-4} Y_{ij,kl}(\eta,\bar{\eta}) - \sum_m C_{ij}{}^{m} C_{mk}{}^{l} |\eta|^{y_i+y_j-y_m-2}\right\}. \end{aligned} \tag{4.22}$$

Note also that expression (4.22) is L independent. In particular, this means that the infrared divergences that were present in individual summands in (4.21) mutually cancel each other. This agrees with the general results of [17, 18][10].

By suitable changes of the integration variable η in the terms with permuted indices i, j, k it is possible to write $\mathcal{F}^{l}_{ijk}$ by means of integrals over three disjoint subsets tiling the whole η-plane. Consider the transformation $\eta \mapsto 1-\eta$, for which the cut-off functions in the integral (4.22) become $\theta(1-|\eta-1|)\theta\big(\mathrm{Re}\,\eta - \frac{1}{2}\big)$. The asymptotics (4.18) for $Y_{ij,kl}$ are such that the divergence of the transformed integrand that arises from the limit $\eta \mapsto 1$ is again cancelled, once we take the transformed subtractions, i.e. the second line in (4.22),

[10] The perturbation expansion for Wilson coefficients proposed in [17, 18] was shown to be IR finite to all orders under certain assumptions on the UV renormalization scheme. As we are interested in scheme-independent quantities their result applies.

into account. The other transformation is $\eta \mapsto 1/\eta$. In this case, the cut-off functions read $\theta(|\eta|-1)\theta(|\eta-1|-1)$ after the transformation, and the divergence of the corresponding integrand for $\eta \to \infty$ is cancelled as well. Together, the regions carved out by the cut-off functions for the three coordinate choices tile the whole η-plane. Using this we can recast (4.22) as

$$\left(\mathcal{F}_{ijk}^{l}\right)_{\text{res}} = \frac{\pi}{3}\int d^2\eta [|\eta|^{2r+y_i+y_j-4}|1-\eta|^{2r+y_j+y_k-4}Y_{ij,kl}(\eta,\bar{\eta}) - S_{ijkl}(\eta)], \tag{4.23}$$

where

$$\begin{aligned} S_{ijkl}(\eta) = {} & \sum_m C_{ij}{}^m C_{mk}{}^l |\eta|^{y_i+y_j-y_m-2}\theta(1-|\eta|)\theta\left(\frac{1}{2}-\operatorname{Re}\eta\right) \\ & + \sum_m C_{kj}{}^m C_{mi}{}^l |1-\eta|^{y_k+y_j-y_m-2}\theta(1-|\eta-1|)\theta\left(\operatorname{Re}\eta-\frac{1}{2}\right) \\ & + \sum_m C_{ik}{}^m C_{mj}{}^l |\eta|^{-y_i-y_k+y_m-2}\theta(|\eta|-1)\theta(|\eta-1|-1). \end{aligned} \tag{4.24}$$

In this form the integration runs over the whole η-plane. Although the subtraction function $S_{ijkl}(\eta)$ still has a piecewise form it is expressed quite explicitly.

4.2.2. Resonant boundary coefficients. On the boundary the computation can be done in a similar way as in the bulk. We consider a boundary perturbation of the form

$$\delta S = \sum_s \mu^s \epsilon^{-y_s}\int dx\, \psi_s(x), \tag{4.25}$$

where now $y_s = 1-h_s$. Up to the third order in the couplings the RG equations take the form

$$\dot{\mu}^s = y_s\mu^s + \sum_{p,q}\mathcal{D}_{pq}^s\mu^p\mu^q + \sum_{p,q,r}\mathcal{G}_{pqr}^s\mu^p\mu^q\mu^r + \cdots. \tag{4.26}$$

As before, we only introduce counterterms at the quadratic order for marginal or relevant fields, and the corresponding coefficients are

$$\mathcal{D}_{pq}^s = D_{pq}{}^s, \tag{4.27}$$

where $D_{pq}{}^s$ is the OPE coefficient of two boundary fields (2.3). In the resonant case where we have $y_s = y_p + y_q + y_r$, the coefficient $\left(\mathcal{G}_{pqr}^s\right)_{\text{res}}$ can be written as

$$\begin{aligned} \left(\mathcal{G}_{pqr}^s\right)_{\text{res}} = {} & \frac{1}{6}\lim_{\epsilon\to 0}\sum_{\text{perm}(p,q,r)}\left\{\epsilon\int_{2\epsilon}^{L}\langle\psi_p(0)\psi_q(\epsilon)\psi_r(x)\psi_s(\infty)\rangle\right. \\ & +\epsilon\int_{-L}^{-\epsilon}\langle\psi_p(x)\psi_q(0)\psi_r(\epsilon)\psi_s(\infty)\rangle - \sum_t \epsilon^{-y_t-y_r+y_s}D_{pq}{}^t\int_{\epsilon}^{L}\langle\psi_t(0)\psi_r(x)\psi_s(\infty)\rangle \\ & \left. -\sum_t \epsilon^{-y_t-y_p+y_s}D_{qr}{}^t\int_{-L}^{-\epsilon}\langle\psi_p(x)\psi_t(0)\psi_s(\infty)\rangle\right\}. \end{aligned} \tag{4.28}$$

By similar arguments as in the previous subsection we can find

$$\begin{aligned} \left(\mathcal{G}_{pqr}^s\right)_{\text{res}} = {} & \frac{1}{6}\int_0^1 d\eta \sum_{\text{perm}(p,q,r)}\left\{\eta^{r+y_p+y_q-2}(1-\eta)^{r+y_q+y_r-2}Y_{pq,rs}(\eta)\right. \\ & \left. -\sum_t D_{pq}{}^t D_{tr}{}^s\eta^{y_p+y_q-y_t-1} - \sum_t D_{qr}{}^t D_{pt}{}^s(1-\eta)^{y_q+y_r-y_t-1}\right\}, \end{aligned} \tag{4.29}$$

where

$$Y_{pq,rs}(\eta) = \sum_t D_{pq}{}^t D_{tr}{}^s F^t_{pq,rs}(\eta) = \sum_t D_{qr}{}^t D_{pt}{}^s F^t_{sp,qr}(1-\eta). \qquad (4.30)$$

Here the conformal blocks $F^t_{pq,rs}(\eta)$ have cuts running from $-\infty$ to 0, and from 1 to $+\infty$. In addition, their asymptotic behaviour is

$$F^t_{pq,rs}(\eta) \sim \eta^{h_t-\delta} \qquad (\eta \to 0), \qquad (4.31)$$

where δ is defined as before, i.e. $\delta = \frac{1}{3}(h_p + h_q + h_r + h_s)$. Finally, the scheme-independent quantity is given by

$$\tilde{\mathcal{G}}^s_{pqr} = \left(\mathcal{G}^s_{pqr}\right)_{\text{res}} + \frac{1}{6} \sum_{\text{perm}(p,q,r)} \sum_t \frac{D_{pq}{}^t \left(D_{tr}{}^s + D_{rt}{}^s\right)}{y_s - y_r - y_t}. \qquad (4.32)$$

In the case where several irreducible boundary conditions are involved, one has to keep track of their labels, and bear in mind the superselection rules, in particular the order of operators. This leads to additional splittings and recombinations of the integrals over four-point functions and subtractions. Apart from this technicality, it is however straightforward to include the boundary labels. We have refrained from writing them explicitly to keep the formulae simpler.

5. Conclusions

In this paper we have studied conformal perturbation theory beyond the leading order. We have shown that, at least up to quadratic order, the combined bulk–boundary perturbation problem is renormalizable, using the minimal subtraction scheme. We also discussed the more commonly used 'Wilsonian' OPE scheme, and found it to have some shortcomings at higher order in perturbation theory. We identified systematically the universal (scheme-independent) quantities, and gave explicit formulae for them at third order in terms of integrals of conformal four-point functions. Finally, we explained how essentially the same analysis works for the pure bulk and pure boundary case. It seems plausible that similar techniques should allow one to prove renormalizability at arbitrary order in perturbation theory, but we have not attempted to do so.

Our work was originally motivated by the question of how the dependence of the conformal dimension of a boundary changing field upon a bulk modulus can be understood from the world-sheet perspective. Our considerations demonstrate that this effect is captured by a certain universal quadratic RG coefficient, for which we gave an explicit formula. This result should also have interesting applications in other contexts; in particular, it provides a world-sheet method to study the stability of brane setups under arbitrary bulk deformations.

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